

# Subspaces intersecting each element of a regulus in one point, André-Bruck-Bose representation and clubs

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## Abstract

In this paper results are proved with applications to the orbits of  $(n - 1)$ -dimensional subspaces disjoint from a regulus  $\mathcal{R}$  of  $(n - 1)$ -subspaces in  $\text{PG}(2n - 1, q)$ , with respect to the subgroup of  $\text{PGL}(2n, q)$  fixing  $\mathcal{R}$ . Such results have consequences on several aspects of finite geometry. First of all, a necessary condition for an  $(n - 1)$ -subspace  $U$  and a regulus  $\mathcal{R}$  of  $(n - 1)$ -subspaces to be extendable to a Desarguesian spread is given. The description also allows to improve results in [4] on the André-Bruck-Bose representation of a  $q$ -subline in  $\text{PG}(2, q^n)$ . Furthermore, the results in this paper are applied to the classification of linear sets, in particular clubs.

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## 1 Introduction

The  $(n - 1)$ -dimensional projective space over the field  $F$  is denoted by  $\text{PG}(n - 1, F)$  or  $\text{PG}(n - 1, q)$  if  $F$  is the finite field of order  $q$  (denoted by  $\mathbb{F}_q$ ). If  $L$  is an extension field  $\mathbb{F}_q$ , then the projective space defined by the  $\mathbb{F}_q$ -vector space induced by  $L^d$  is denoted by  $\text{PG}_q(L^d)$ . For further notation and general definitions employed in this paper the reader is referred to [9, 11, 13]. For more information on Desarguesian spreads see [1].

This paper is structured as follows. In Section 2 subspaces which intersect each element of a regulus in one point are studied and a result from [6] is generalised. Section 3 contains one of the main results of this paper, determining the order of the normal rational curves obtained from  $n$ -dimensional subspaces on an external  $(n - 1)$ -dimensional subspace with respect to a regulus in  $\text{PG}(2n - 1, q)$ , obtained from a point and a subline after applying the field reduction map to  $\text{PG}(1, q^n)$ . This leads to a necessary condition on the existence of a Desarguesian spread containing a subspace and regulus (Corollary 3.4). The André-Bruck-Bose representation of

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sublines and subplanes of a finite projective plane is studied in Section 4 and improvements are obtained with respect to the known results [5, 14, 15, 4]. The results from the first sections are then applied to the classification problem for clubs of rank three in  $\text{PG}(1, q^n)$  in Section 5. A study of the incidence structure of the clubs in  $\text{PG}(1, q^n)$  after field reduction yields to a partial classification, concluding that the orbits of clubs under  $\text{PGL}(2, q^n)$  are at least  $k - 1$ , where  $k$  stands for the number of divisors of  $n$ . The paper concludes with an appendix discussing a result motivated by Burau [6] for the complex numbers: the result is extended to general algebraically closed fields; a new proof is provided; and counterexamples are given to some of the arguments used in the original proof.

## 2 Subspaces intersecting each element of a regulus in one point

Let  $\mathcal{R}$  be a regulus of subspaces in a projective space and let  $S$  be any subspace of  $\langle \mathcal{R} \rangle$ . Questions about the properties of the set of intersection points, which for reasons of simplicity of notation we will denote by  $S \cap \mathcal{R}$ , often turn up while investigating objects in finite geometry. If  $S$  intersects each element of the regulus  $\mathcal{R}$  in a point, then the intersection  $S \cap \mathcal{R}$  is a normal rational curve, see Lemma 2.1. This was already pointed out in [6, p.173] with a proof originally intended for complex projective spaces, but actually holding in a more general setting. The notation of [6] will be partly adopted.

The Segre variety representing the Cartesian product  $\text{PG}(n, F) \times \text{PG}(m, F)$  in  $\text{PG}((n+1)(m+1) - 1, F)$  is denoted by  $\mathcal{S}_{n,m,F}$ . It is well known that  $\mathcal{S}_{n,m,F}$  contains two families  $\mathcal{S}_{n,m,F}^I$  and  $\mathcal{S}_{n,m,F}^{II}$  of maximal subspaces of dimensions  $n$  and  $m$ , respectively. When convenient, the notation  $S^I$  or  $S^{II}$  will be used for a subspace belonging to the first or second family. The points of  $\mathcal{S}_{n,m,F}$  may be represented as one-dimensional subspaces spanned by rank one  $(m+1) \times (n+1)$  matrices. This is the standard example of a regular embedding of product spaces, see [16]. Note that in the finite case it is possible to embed product spaces in projective spaces of smaller dimension (see e.g. [7]). A regulus  $\mathcal{R}$  of  $(n-1)$ -dimensional subspaces can also be defined as  $\mathcal{S}_{n-1,1,F}^I$ .

**LEMMA 2.1.** *Let  $n > 1$  be an integer, and  $F$  a field. Let  $S_t$  be a  $t$ -subspace of  $\text{PG}(2n-1, F)$  intersecting each  $S^I \in \mathcal{S}_{n-1,1,F}^I$  in precisely one point. Define  $\Phi = S_t \cap \mathcal{S}_{n-1,1,F}$ , and assume  $\langle \Phi \rangle = S_t$ . Then  $|F| \geq t$  and the following properties hold.*

- (i) *The set  $\Phi$  is a normal rational curve of order  $t$ .*
- (ii) *Let  $\Xi^I \in \mathcal{S}_{n-1,1,F}^I$ . Then the set  $S(\Phi, \Xi^I)$  of the intersections of  $\Xi^I$  with all transversal lines  $l^{II}$  such that  $l^{II} \cap \Phi \neq \emptyset$  is a normal rational curve of order  $t$  or  $t-1$  if  $|F| = t$ , and of order  $t-1$  if  $|F| > t$ .*
- (iii) *If  $\Phi$  is contained in a subvariety  $\mathcal{S}_{t-1,1,F}$  of  $\mathcal{S}_{n-1,1,F}$ , then homogeneous coordinates can be chosen such that  $\Phi$  is represented parametrically by*

$$\left\langle \begin{pmatrix} y_0^t & y_0^{t-1}y_1 & \cdots & y_0y_1^{t-1} \\ y_0^{t-1}y_1 & y_0^{t-2}y_1^2 & \cdots & y_1^t \end{pmatrix} \right\rangle, \quad (y_0, y_1) \in (F^2)^*, \quad (1)$$

*and  $S(\Phi, \Xi^I)$ , for  $z_0, z_1$  depending only on  $\Xi^I$ , by*

$$\left\langle \begin{pmatrix} y_0^{t-1}z_0 & y_0^{t-2}y_1z_0 & \cdots & y_1^{t-1}z_0 \\ y_0^{t-1}z_1 & y_0^{t-2}y_1z_1 & \cdots & y_1^{t-1}z_1 \end{pmatrix} \right\rangle, \quad (y_0, y_1) \in (F^2)^*. \quad (2)$$

*Proof.* (i), (iii) The proof in [6, Sect.41 no.3], which is offered for  $F = \mathbb{C}$ , works exactly the same provided that  $|F| > t$  or, more generally, that  $\Phi$  is contained in some subvariety  $\mathcal{S}_{t-1,1,F}$  of  $\mathcal{S}_{n-1,1,F}$ . In case  $|F| \leq t$ , the size of  $\Phi$  being  $|F| + 1$  implies  $|F| = t$ , so  $\Phi$  is just a set of  $t + 1$  independent points in a subspace isomorphic to  $\text{PG}(t, t)$ , hence  $\Phi$  is a normal rational curve of order  $t$ .

(ii) The case  $|F| > t$  is proved in [6] immediately after the corollary at p. 175. If  $|F| \leq t$ , then  $|F| = t$  and two cases are possible. If  $\Phi$  is contained in some  $\mathcal{S}_{t-1,1,F} \subseteq \mathcal{S}_{n-1,1,F}$ , Burau's proof is still valid as was mentioned in case (ii); so,  $S(\Phi, \Xi^I)$  is a normal rational curve of order  $t - 1 = |F| - 1$ . Otherwise  $S(\Phi, \Xi^I)$  is an independent  $(t + 1)$ -set, hence a normal rational curve of order  $|F|$ .  $\square$

REMARK 2.2. If  $|F| = t$  both cases in Lemma 2.1 (ii) can occur. The following two examples use the Segre embedding  $\sigma = \sigma_{t-1,1,F}$  of the product space  $\text{PG}(t-1, t) \times \text{PG}(1, t)$  in  $\text{PG}(2t-1, t)$ . Let  $\{s_0, s_1, \dots, s_t\}$  be the set of points on  $\text{PG}(1, t)$  and suppose  $\{r_0, r_1, \dots, r_t\}$  is a set of  $t + 1$  points in  $\text{PG}(t-1, t)$ . Put  $\Xi^I = \sigma(\text{PG}(1, t) \times s_0)$  and  $\Phi := \{\sigma(r_i \times s_i) : i = 0, 1, \dots, t\}$ . Then  $\Phi$  consists of  $t + 1$  points on the Segre variety  $\mathcal{S}_{t-1,1,F}$ . Depending on the set  $\{r_0, r_1, \dots, r_t\}$  one obtains the two cases described in Lemma 2.1 (ii).

- a. If  $\{r_0, r_1, \dots, r_t\}$  is a frame of a hyperplane of  $\text{PG}(t-1, t)$  then  $\Phi$  generates a  $t$ -dimensional subspace of  $\text{PG}(2t-1, t)$  intersecting  $\mathcal{S}_{t-1,1,F}$  in  $\Phi$  and  $S(\Phi, \Xi^I)$  is a normal rational curve of order  $t - 1$ .
- b. If  $\{r_0, r_1, \dots, r_t\}$  generates  $\text{PG}(t-1, t)$  then  $\Phi$  generates a  $t$ -dimensional subspace of  $\text{PG}(2t-1, t)$  intersecting  $\mathcal{S}_{t-1,1,F}$  in  $\Phi$  and  $S(\Phi, \Xi^I)$  is a normal rational curve of order  $t$ .

REMARK 2.3. By (1) and (2), the map  $\alpha : \Phi \rightarrow S(\Phi, \Xi^I)$  defined by the condition that  $X$  and  $X^\alpha$  are on a common line in  $\mathcal{S}_{n-1,1,F}^{II}$  is related to a projectivity between the parametrizing projective lines. Such an  $\alpha$  is also called a *projectivity*.

### 3 The order of normal rational curves contained in $\mathcal{S}_{n-1,1,q}$

Here  $n \geq 2$  is an integer. The field reduction map  $\mathcal{F}_{m,n,q}$  from  $\text{PG}(m-1, q^n)$  to  $\text{PG}(mn-1, q)$  will also be denoted by  $\mathcal{F}$ . If  $S$  is a set of points, in  $\text{PG}(m-1, q^n)$ , then  $\mathcal{F}(S)$  is a set of subspaces, whose union, as a set of points will be denoted by  $\tilde{\mathcal{F}}(S)$ . The  $\mathbb{F}_{q^h}$ -span of a subset  $b$  of  $\text{PG}(d, q^n)$  is denoted by  $\langle b \rangle_{q^h}$ .

PROPOSITION 3.1. *Let  $b$  be a  $q$ -subline of  $\text{PG}(1, q^n)$ , and let  $\Theta \notin b$  be a point of  $\text{PG}(1, q^n)$ . Let  $1, \zeta$  and  $1, \zeta'$  be homogeneous coordinates of  $\Theta$  with respect to two reference frames for  $\langle b \rangle_{q^n}$ , each of which consists of three points of  $b$ . Then  $\mathbb{F}_q(\zeta) = \mathbb{F}_q(\zeta')$ .*

*Proof.* Homogeneous coordinates of a point in both reference frames, say  $(x_0, x_1)$  and  $(x'_0, x'_1)$ , are related by an equation of the form  $\rho(x'_0 \ x'_1)^T = A(x_0 \ x_1)^T$ ,  $\rho \in \mathbb{F}_{q^n}^*$ ,  $A \in \text{GL}(2, q)$ . Hence  $(\rho \ \rho\zeta')^T = A(1 \ \zeta)^T$  and this implies  $\zeta' \in \mathbb{F}_q(\zeta)$ . The proof of  $\zeta \in \mathbb{F}_q(\zeta')$  is similar.  $\square$

By Proposition 3.1, the *degree of a point over a  $q$ -subline  $b$*  in a finite projective space  $\text{PG}(d, q^n)$ ,  $[\Theta : b] = [\mathbb{F}_q(\zeta) : \mathbb{F}_q]$  for  $\Theta \in \langle b \rangle_{q^n} \setminus b$ ,  $[\Theta : b] = 1$  for  $\Theta \in b$ , is well-defined. This  $[\Theta : b]$  also equals the minimum integer  $m$  such that a subgeometry  $\Sigma \cong \text{PG}(d, q^m)$  exists containing both  $b$  and  $\Theta$ .

PROPOSITION 3.2. Any  $n$ -subspace of  $\text{PG}(2n-1, q)$  containing an  $(n-1)$ -subspace  $S^I \in \mathcal{S}_{n-1,1,q}^I$  intersects  $\mathcal{S}_{n-1,1,q}$  in the union of  $S^I$  and a line in  $\mathcal{S}_{n-1,1,q}^{II}$ .

THEOREM 3.3. Let  $b$  be a  $q$ -subline of  $\text{PG}(1, q^n)$ , and  $\Theta \notin b$  a point of  $\text{PG}(1, q^n)$ . Then in  $\text{PG}(2n-1, q)$  any  $n$ -subspace  $\mathcal{H}$  containing  $\mathcal{F}(\Theta)$  intersects the Segre variety  $\mathcal{S}_{n-1,1,q} = \tilde{\mathcal{F}}(b)$ , in a normal rational curve whose order is  $\min\{q, [\Theta : b]\}$ .

*Proof.* Set  $L = \mathbb{F}_{q^n}$ ,  $F = \mathbb{F}_q$ . Without loss of generality,  $\text{PG}(2n-1, q) = \text{PG}_q(L^2)$ ,  $\mathcal{F}(b) = \{L(x, y) \mid (x, y) \in (F^2)^*\}^1$ , and  $\Theta = L(1, \xi)$  with  $[F(\xi) : F] = [\Theta : b]$ . The  $n$ -subspace  $\mathcal{H}$  intersects  $L(1, 0)$  in one point  $Y$  of the form  $Y = F(\theta, 0)$ ,  $\theta \in L^*$ . For any  $x \in F$ , seeking for the intersection  $\langle \mathcal{F}(\Theta), Y \rangle_q \cap L(x, 1)$ , or

$$\langle L(1, \xi), F(\theta, 0) \rangle_q \cap L(x, 1)$$

gives two equations in  $\alpha, \beta \in L$ :

$$\alpha + \theta = \beta x, \quad \alpha \xi = \beta,$$

whence  $\beta = \theta(x - \xi^{-1})^{-1}$ . The intersection point is then  $F(x\theta(x - \xi^{-1})^{-1}, \theta(x - \xi^{-1})^{-1})$ . So, for  $\Xi = L(0, 1)$ , the set of the intersections of  $\Xi$  with all lines in  $\mathcal{S}_{n-1,1,q}^{II}$  which meet  $\mathcal{H}$  is

$$S(\mathcal{H} \cap \mathcal{S}_{n-1,1,q}, \Xi) = \{F(0, \theta(x - \xi^{-1})^{-1}) \mid x \in \mathcal{F}_q\} \cup \{F(0, \theta)\}.$$

This  $S(\mathcal{H} \cap \mathcal{S}_{n-1,1,q}, \Xi)$  is obtained by inversion from the line joining the points  $F(0, \theta^{-1})$  and  $F(0, \theta^{-1}\xi^{-1})$ . By [10, Theorem 5],  $\mathcal{C}_Y$  is a normal rational curve of order  $\delta' = \min\{q, [F(\xi^{-1}) : F] - 1\} = \min\{q, [\Theta : b] - 1\}$ . Now apply lemma 2.1 for  $S_t = \langle \mathcal{H} \cap \mathcal{S}_{n-1,1,q} \rangle_q$ : if  $t \geq q$ , then  $t = q$  and  $\delta' = q$  or  $\delta' = q - 1$ , so  $[\Theta : b] \geq q$  and  $t = \min\{q, [\Theta : b]\}$ . If on the contrary  $t < q$ , then  $t - 1 = \delta' = [\Theta : b] - 1$ , so  $t = [\Theta : b]$  and  $t = \min\{q, [\Theta : b]\}$  again.  $\square$

An important consequence of the above result answers the question of the existence of a Desarguesian spread containing a given regulus  $\mathcal{R}$  and a subspace disjoint from  $\mathcal{R}$ .

COROLLARY 3.4. If a regulus  $\mathcal{R} = \mathcal{S}_{n-1,1,q}$  and an  $(n-1)$ -dimensional subspace  $U$ , disjoint from  $\mathcal{R}$ , in  $\text{PG}(2n-1, q)$  are contained in a Desarguesian spread then there is an integer  $c$  such that any  $n$ -subspace  $\mathcal{H}$  containing  $U$  intersects  $\mathcal{R}$  in a normal rational curve of order  $c$ .

The following remark illustrates that this necessary condition is not always satisfied.

REMARK 3.5. For  $n > 2$  by using the package *FinInG* [2] of *GAP* [3] examples can be given of  $(n-1)$ -subspaces disjoint from  $\mathcal{S}_{n-1,1,q}$  contained in  $n$ -subspaces intersecting the Segre variety in normal rational curves of distinct orders. We include one explicit example. Let  $q = 4$ ,  $\mathbb{F}_q = \mathbb{F}_2(\omega)$ , with  $\omega^2 + \omega + 1 = 0$ . Let  $\mathcal{R}$  be the regulus of 3-dimensional subspaces of  $\text{PG}(7, 4)$  obtained from the standard subline  $\text{PG}(1, q)$  in  $\text{PG}(1, q^4)$ , and put

$$S_3 := \langle (1, 0, 0, 0, \omega^2, 1, 0, 1), (0, 1, 0, 0, 1, \omega^2, 0, \omega^2), (0, 0, 1, 0, 0, \omega, 1, \omega), (0, 0, 0, 1, \omega^2, \omega^2, \omega, 1) \rangle.$$

Then  $S_3$  is a three-dimensional subspace disjoint from the regulus  $\mathcal{R}$ . Moreover, the 4-dimensional subspace  $\langle S_3, (1, 0, 0, 0, 0, 0, 0, 0) \rangle$  intersects the regulus  $\mathcal{R}$  in a normal rational curve of degree 4, while the 4-dimensional subspace  $\langle S_3, (0, 1, 0, \omega^2, 0, 0, 0, 0) \rangle$  intersects  $\mathcal{R}$  in a conic.

<sup>1</sup>For  $x, y \in L$ ,  $F(x, y) = \langle (x, y) \rangle_q$ , and  $L(x, y) = \langle (x, y) \rangle_{q^n}$ .

## 4 André-Bruck-Bose representation

The André-Bruck-Bose representation of a Desarguesian affine plane of order  $q^n$  is related to the image of  $\text{PG}(2, q^n)$ , under the field reduction map  $\mathcal{F}$ , by means of the following straightforward result.

**PROPOSITION 4.1.** *Let  $\mathcal{D}$  be the Desarguesian spread in  $\text{PG}(3n-1, q)$  obtained after applying the field reduction map  $\mathcal{F}$  to the set of points of  $\text{PG}(2, q^n)$ ,  $l_\infty$  a line in  $\text{PG}(2, q^n)$ , and  $\mathcal{K}$  a  $(2n)$ -subspace of  $\text{PG}(3n-1, q)$ , containing the spread  $\mathcal{F}(l_\infty)$ . Take  $\text{PG}(2, q^n) \setminus l_\infty$  and  $\mathcal{K} \setminus \langle \mathcal{F}(l_\infty) \rangle_q$  as representatives of  $\text{AG}(2, q^n)$  and  $\text{AG}(2n, q)$ , respectively. Then the map  $\varphi : \text{AG}(2, q^n) \rightarrow \text{AG}(2n, q)$  defined by  $\varphi(X) = \mathcal{F}(X) \cap \mathcal{K}$  for any  $X \in \text{AG}(2, q^n)$  is a bijection, mapping lines of  $\text{AG}(2, q^n)$  into  $n$ -subspaces of  $\text{AG}(2n, q)$  whose  $(n-1)$ -subspaces at infinity belong to the spread  $\mathcal{F}(l_\infty)$ .*

The notation in Proposition 4.1 is assumed to hold in the whole section. The following result improves [4, Theorems 3.3 and 3.5], by determining the order of the involved normal rational curves.

**THEOREM 4.2.** *Let  $b$  be a  $q$ -subline of  $\text{PG}(2, q^n)$ , not contained in  $l_\infty$ . Set  $\Theta = \langle b \rangle_{q^n} \cap l_\infty$ . Then the André-Bruck-Bose representation  $\varphi(b \setminus l_\infty)$  is the affine part of a normal rational curve whose order is  $\delta = \min\{q, [\Theta : b]\}$ . More precisely, if  $\delta = 1$ , then  $\varphi(b \setminus l_\infty)$  is an affine line; if  $\delta > 1$ , then  $b \cap l_\infty = \emptyset$ , and  $\varphi(b)$  is a normal rational curve with no points at infinity.*

*Proof.* The intersection  $\mathcal{H} = \langle \mathcal{F}(b) \rangle_q \cap \mathcal{K}$  is an  $n$ -space containing  $\mathcal{F}(\Theta)$ , and contained in the span of the Segre variety  $\mathcal{S}_{n-1,1,q} = \mathcal{F}(b)$ . The result follows from Proposition 3.2 and Theorem 3.3.  $\square$

The results in [4, Theorems 3.3 and 3.5] also characterize the normal rational curves arising from  $q$ -sublines in  $\text{AG}(2, q^n)$ .

In [5, 14, 15] for  $n = 2$  and [4, Theorem 3.6 (a)(b)] for any  $n$  the André-Bruck-Bose representation of a  $q$ -subplane tangent to a line at the infinity is described. Further properties are stated in the following theorem:

**THEOREM 4.3.** *Let  $B$  be a  $q$ -subplane of  $\text{PG}(2, q^n)$  that is tangent to  $l_\infty$  at the point  $T$ . Let  $b$  be a line of  $B$  not through  $T$ ,  $\Theta = \langle b \rangle_{q^n} \cap l_\infty$ , and  $\delta = \min\{q, [\Theta : b]\}$ . Then there are a normal rational curve  $\mathcal{C}_0$  of order  $\delta$  in the  $n$ -subspace  $\varphi(\langle b \rangle_{q^n})$ , a normal rational curve  $\mathcal{C}_1 \subset \mathcal{F}(T)$  of order  $\delta'$ , with*

$$\delta' \begin{cases} = [\Theta : b] - 1 & \text{for } q > [\Theta : b] \\ \in \{q-1, q\} & \text{otherwise,} \end{cases} \quad (3)$$

*and a projectivity  $\kappa : \mathcal{C}_0 \rightarrow \mathcal{C}_1$  (in the sense of Remark 2.3), such that  $\varphi(B \setminus l_\infty)$  is the ruled surface union of all lines  $XX^\kappa$  for  $X \in \mathcal{C}_0$ .*

*Proof.* By Theorem 4.2,  $\mathcal{C}_0 := \varphi(b)$  is a normal rational curve of order  $\delta$  in the  $n$ -subspace  $\varphi(\langle b \rangle_{q^n} \setminus l_\infty)$ , and for any  $P = \varphi(X) \in \mathcal{C}_0$ , the subline  $TX$  of  $B$  corresponds to an affine line  $PP^\kappa$  with  $P^\kappa \in \mathcal{F}(T)$  at infinity. Define  $\mathcal{C}_1 = \{P^\kappa \mid P \in \mathcal{C}_0\}$ .

By the field reduction map  $\mathcal{F} = \mathcal{F}_{3,n,q}$ , the subplane  $B$  is mapped to  $\mathcal{F}(B)$  which is the set of all maximal subspaces of the first family in  $\mathcal{S}_{n-1,2,q} \subset \text{PG}(3n-1, q)$ . The vector homomorphism

$$(\lambda, v) \in \mathbb{F}_{q^n} \times \mathbb{F}_q^3 \mapsto \lambda \otimes_{\mathbb{F}_q} v$$

corresponds to a projective embedding  $g : \text{PG}(n-1, q) \times B \rightarrow \mathcal{S}_{n-1,2,q}$  whose image is  $\mathcal{S}_{n-1,2,q}$ , and such that  $\mathcal{F}(X) = (\text{PG}(n-1, q) \times X)^g$  for any point  $X$  in  $B$ . It holds  $\varphi(B \setminus l_\infty) = \mathcal{S}_{n-1,2,q} \cap \mathcal{K} \setminus \mathcal{F}(T)$ . For any point  $U$  in  $B$  define

$$\kappa_U : (X, Y)^g \in \mathcal{S}_{n-1,2,q} \mapsto (X, U)^g \in \mathcal{F}(U).$$

Note that for any  $Y \in B$ , the restriction of  $\kappa_U$  to  $\mathcal{F}(Y)$  is a projectivity. For any  $U \in b$ , using the notation from Lemma 2.1 it holds  $\mathcal{C}_0^{\kappa_U} = S(\mathcal{C}_0, \mathcal{F}(U))$ , and as a consequence,  $\mathcal{C}_0^{\kappa_U}$  is a normal rational curve of order  $\delta'$  as in (3). Now, since for any  $P \in \mathcal{C}_0$ , say  $P = (X_P, Y_P)^g$ , the points  $P$ ,  $P^\kappa$  and  $P^{\kappa_T}$  are on the plane  $(X_P \times B)^g \in \mathcal{S}_{n-1,2,q}^{II}$ , and  $P^\kappa, P^{\kappa_T} \in \mathcal{F}(T)$ , it follows that  $P^\kappa = P^{\kappa_T}$ . It also follows that  $\mathcal{C}_1 = \mathcal{C}_0^{\kappa_U \kappa_T} = S(\mathcal{C}_0, \mathcal{F}(U))^{\kappa_T}$ , and hence  $\mathcal{C}_1$  is a normal rational curve of order  $\delta'$  as in (3). Finally,  $\kappa_U : \mathcal{C}_0 \rightarrow S(\mathcal{C}_0, \mathcal{F}(U))$  is a projectivity as defined in Remark 2.3, and hence so is  $\kappa$ .  $\square$

## 5 On the classification of clubs

An  $\mathbb{F}_q$ -club (or simply a club) in  $\text{PG}(1, q^n)$  is an  $\mathbb{F}_q$ -linear set of rank three, having a point of weight two, called the *head* of the club. An  $\mathbb{F}_q$ -club has  $q^2 + 1$  points, and the non-head points have weight one. From now on it will be assumed that  $n > 2$ . The next proposition is a straightforward consequence of the representation of linear sets as projections of subgeometries [12, Theorem 2].

**PROPOSITION 5.1.** *Let  $L$  be an  $\mathbb{F}_q$ -club in  $\text{PG}(1, q^n) \subset \text{PG}(2, q^n)$ . Then there are a  $q$ -subplane  $\Sigma$  of  $\text{PG}(2, q^n)$ , a  $q$ -subline  $b$  in  $\Sigma$ , and a point  $\Theta \in \langle b \rangle_{q^n} \setminus b$ , such that  $L$  is the projection of  $\Sigma$  from the center  $\Theta$  onto the axis  $\text{PG}(1, q^n)$ .*

As before the notation  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  is used, where  $\mathcal{F} = \mathcal{F}_{2,n,q}$  denotes the field reduction map from  $\text{PG}(1, q^n)$  to  $\text{PG}(2n-1, q)$ .

**PROPOSITION 5.2.** *Let  $L$  be an  $\mathbb{F}_q$ -club of  $\text{PG}(1, q^n)$  with head  $\Upsilon$ . Then  $\tilde{\mathcal{F}}(L)$  contains two collections of subspaces, say  $F_1$  and  $F_2$ , satisfying the following properties.*

- (i) *The subspaces in  $F_1$  are  $(n-1)$ -dimensional, are pairwise disjoint, and any subspace in  $F_1$  is disjoint from  $\mathcal{F}(\Upsilon)$ .*
- (ii) *Any subspace in  $F_2$  is a plane and intersects  $\mathcal{F}(\Upsilon)$  in precisely a line.*
- (iii) *Any point of  $\mathcal{F}(\Upsilon)$  belongs to exactly  $q+1$  planes in  $F_2$ .*
- (iv) *If  $L$  is not isomorphic to  $\text{PG}(1, q^2)$ , and  $l$  is any line of  $\text{PG}(2n-1, q)$  contained in  $\tilde{\mathcal{F}}(L)$ , then  $l$  is contained in  $\mathcal{F}(\Upsilon)$  or in a subspace in  $F_1 \cup F_2$ .*

*Proof.* The assumptions imply the existence of  $\Sigma$  and a  $q$ -subline  $b$  in  $\Sigma$  as in Proposition 5.1. The assertions are a consequence of the fact that  $\tilde{\mathcal{F}}(\Sigma)$  is a Segre variety  $\mathcal{S}_{n-1,2,q}$  in  $\text{PG}(3n-1, q)$ . Let

$$p_1 : \text{PG}(2, q^n) \setminus \Theta \rightarrow \text{PG}(1, q^n)$$

be the projection with center  $\Theta$ , associated with

$$p_2 : \text{PG}(3n-1, q) \setminus \mathcal{F}(\Theta) \rightarrow \text{PG}(2n-1, q).$$

The collections  $F_1$  and  $F_2$  are defined as follows:

$$F_1 = \{\mathcal{F}(p_1(X)) \mid X \in \Sigma \setminus b\} = \mathcal{F}(L) \setminus \mathcal{F}(\Upsilon), \quad F_2 = \{p_2(V^{II}) \mid V^{II} \in \tilde{\mathcal{F}}(\Sigma)^{II}\}.$$

The assertion (i) is straightforward, as well as  $\dim(V) = 2$  for any  $V \in F_2$ . For any  $V^{II} \in \tilde{\mathcal{F}}(\Sigma)^{II}$ , the intersection  $V^{II} \cap \langle \tilde{\mathcal{F}}(b) \rangle_q$  is a line, and this with  $p_2^{-1}(\mathcal{F}(\Upsilon)) = \langle \tilde{\mathcal{F}}(b) \rangle_q \setminus \mathcal{F}(\Theta)$  implies the second assertion in (ii). Next, let  $P$  be a point in  $\mathcal{F}(\Upsilon)$ . A plane  $V = p_2(V^{II})$  contains  $P$  if, and only if,  $V^{II}$  intersects the  $n$ -subspace  $\langle \mathcal{F}(\Theta), P \rangle_q$ , that is,  $V^{II}$  intersects the normal rational curve  $\mathcal{S}_{n-1,2,q} \cap \langle \mathcal{F}(\Theta), P \rangle_q$ ; this implies (iii).

Assume that a line  $l \subset \tilde{\mathcal{F}}(L)$  exists which is neither contained in  $\mathcal{F}(\Upsilon)$ , nor in a  $T \in F_1 \cup F_2$ . Let  $Q$  be a point in  $l \setminus \mathcal{F}(\Upsilon)$ , and let  $V \in F_2$  such that  $Q \in V$ . It holds  $L = \mathcal{B}(V)$ . Then  $\mathcal{B}(l)$  is a  $q$ -subline of  $L$ . Suppose that a line  $l'$  in  $V$  exists such that  $\mathcal{B}(l') = \mathcal{B}(l)$ . Since  $\mathcal{B}(Q) \neq \mathcal{B}(Q')$  for any  $Q' \in V$ ,  $Q' \neq Q$ , the line  $l'$  contains  $Q$ . Then  $l, l'$  are two distinct transversal lines in  $\mathcal{B}(l)^{II}$ , a contradiction. Hence  $\mathcal{B}(l') \neq \mathcal{B}(l)$  for any line  $l'$  in  $V$ , that is,  $\mathcal{B}(l)$  is a so-called *irregular subline* [8]. By [8, Corollary 13], no irregular subline exists in  $L$ , and this contradiction implies (iv).  $\square$

**PROPOSITION 5.3.** *Let  $L$  be an  $\mathbb{F}_q$ -club with head  $\Upsilon$ . Let  $\Theta$  be the point and  $b$  be the subline as defined in Proposition 5.1. Then for any point  $X$  in  $\mathcal{F}(\Upsilon)$ , the intersection lines of  $\mathcal{F}(\Upsilon)$  with any  $q$  distinct planes in  $F_2$  containing  $X$  span an  $s$ -dimensional subspace, where*

$$(i) \quad s = [\Theta : b] - 1 \text{ if } q > [\Theta : b];$$

$$(ii) \quad s \in \{q - 1, q\} \text{ if } q \leq [\Theta : b].$$

*Proof.* Let  $p_2$  be the projection map as defined in the proof of Proposition 5.2,  $X = p_2(P)$ , and  $\mathcal{H} = \langle \mathcal{F}(\Theta), P \rangle_q$ . For any plane  $V = p_2(V^{II})$ , it holds  $X \in V$  if, and only if  $V^{II} \cap \mathcal{H} \neq \emptyset$ . The intersection  $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$  is a normal rational curve of order  $\min\{q, [\Theta : b]\}$  (cf. Theorem 3.3). Let  $V_0 = p_2(V_0^{II})$  be the unique plane of  $F_2$  through  $X$  distinct from the  $q$  planes chosen in the assumptions (cf. Proposition 5.2). Let  $Q = \tilde{\mathcal{F}}(b) \cap V_0^{II}$ ;  $\mathcal{B}(Q)$  is an  $(n - 1)$ -subspace of  $\tilde{\mathcal{F}}(b)^I$ . Such  $\mathcal{B}(Q)$  is mapped onto  $\mathcal{B}(X) = \mathcal{F}(\Upsilon)$  by  $p_2$ . Assume  $V_i = p_2(V_i^{II})$ ,  $i = 1, 2, \dots, q$ , are the  $q$  planes chosen in the assumptions. Any  $V_i^{II}$ ,  $i = 1, 2, \dots, q$ , intersects  $\mathcal{H}$ , hence  $V_i^{II} \cap \mathcal{B}(Q)$  is the intersection of  $\mathcal{B}(Q)$  with a transversal line of  $\tilde{\mathcal{F}}(b)$  intersecting the normal rational curve  $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$ . By Lemma 2.1 (ii), the set

$$S = \{V_i^{II} \cap \mathcal{B}(Q) \mid i = 1, 2, \dots, q\} \cup \{Q\}$$

is a normal rational curve of order  $s$  where  $s$  takes the values as stated in (i) and (ii). Since  $V_i \cap \mathcal{F}(\Upsilon)$  is the line through  $X$  and a point of  $p_2(S)$ , distinct from  $X$ , the span of the intersection lines is the same as the span of  $p_2(S)$ .  $\square$

**THEOREM 5.4.** *Let  $\mathcal{I}_{n,q}$  be the set of integers  $h$  dividing  $n$  and such that  $1 < h < q$ . For any  $h \in \mathcal{I}_{n,q}$ , let  $L_h$  be the linear set obtained by projecting a  $q$ -subplane  $\Sigma$  of  $\text{PG}(2, q^n)$  from a point  $\Theta_h$  collinear with a  $q$ -subline  $b$  in  $\Sigma$  and such that  $[\Theta_h : b] = h$ . Then the set  $\Lambda = \{L_h \mid h \in \mathcal{I}_{n,q}\}$  contains  $\mathbb{F}_q$ -clubs in  $\text{PG}(1, q^n)$  all belonging to distinct orbits under  $\text{PGL}(2, q^n)$ .*

*Proof.* If  $n$  is odd, then no club is isomorphic to  $\text{PG}(1, q^2)$ . So, by Proposition 5.2 (iv), the families  $F_1$  and  $F_2$  are uniquely determined. The thesis is a consequence of Proposition 5.3, taking into account that if  $L$  and  $L'$  are projectively equivalent, then  $\tilde{\mathcal{F}}(L)$  and  $\tilde{\mathcal{F}}(L')$  are projectively equivalent in  $\text{PG}(2n - 1, q)$ .

In order to deal with the case  $n$  even, it is enough to show that in  $\Lambda$  at most one club is isomorphic to  $\text{PG}(1, q^2)$ . So assume  $L_h \cong \text{PG}(1, q^2)$ . Then  $\tilde{\mathcal{F}}(L_h)$  has a partition  $\mathcal{P}_1$  in  $(n-1)$ -subspaces, and a partition  $\mathcal{P}_2$  in 3-subspaces. From [8, Lemma 11] it can be deduced that any line contained in  $\tilde{\mathcal{F}}(L_h)$  is contained in an element of  $\mathcal{P}_1$  or  $\mathcal{P}_2$ . The intersections of a subspace  $U$  of a family  $\mathcal{P}_i$  with the elements of the other family form a line spread of  $U$ . Hence all planes in  $F_2$  are contained in 3-subspaces of  $\mathcal{P}_2$ , and all planes of  $F_2$  through a point  $X$  in  $\mathcal{F}(\Upsilon)$  meet  $\mathcal{F}(\Upsilon)$  in the same line. By Proposition 5.3 this implies  $h = 2$ .  $\square$

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## A Appendix: On a result in [6]

In [6, p.175] the following result (*Korollar*) is stated for  $F = \mathbb{C}$ .

**COROLLARY A.1.** *Let  $F$  be an algebraically closed field. If an  $s$ -subspace  $S_s$  of  $\text{PG}(2s-1, F)$  meets all  $S^I \in \mathcal{S}_{s-1,1,F}^I$  only in points, then such points span  $S_s$ .*

In [6] the previous result is seemingly proved using methods valid in any field with enough elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the authors the proof in [6] is obtained using an erroneous argument. As a matter of fact, it is claimed in the proof at page 174 that the assumption  $\langle \Phi \rangle = S_s$  is not used. However the contradiction  $S_s \subset \langle \mathcal{S}_{s-2,1,\mathbb{C}} \rangle$  is inferred from  $\Phi \subset \mathcal{S}_{s-2,1,\mathbb{C}}$ .

A further counterexample, which exists whenever a hyperbolic quadric  $Q^+(3, F)$  in a three-dimensional projective space admits an external line (a condition which is not met when the field  $F$  is algebraically closed) is the following. If  $\ell$  is the line corresponding to the two-dimensional vector space  $\langle e_1 \rangle \otimes \langle e'_1, e'_2 \rangle$  and  $m$  is a line external to the hyperbolic quadric obtained by the intersection of the Segre variety  $\mathcal{S}_{2,1,F}$  with the 3-space corresponding to the vector space  $\langle e_2, e_3 \rangle \otimes \langle e'_1, e'_2 \rangle$ , then the 3-dimensional subspace  $\langle \ell, m \rangle$  intersects  $\mathcal{S}_{2,1,F}$  in the line  $\ell$  belonging to  $\mathcal{S}_{2,1,F}^{II}$ .

For the sake of completeness, a proof for corollary A.1 is given.

*Proof of corollary A.1.* Define

$$S_t = \langle S_s \cap \mathcal{S}_{s-1,1,F} \rangle, \quad t = \dim S_t \tag{4}$$

and suppose  $t < s$ . It is proved in [6, p.173 (6)] that  $S_t \subset \langle \mathcal{S}_{t-1,1,F} \rangle$  for some  $\mathcal{S}_{t-1,1,F} \subset \mathcal{S}_{s-1,1,F}$ .

Note that  $S_s \cap \langle \mathcal{S}_{t-1,1,F} \rangle = S_t$ ; otherwise, comparing dimensions,  $S_s$  would intersect each  $S^I \in \mathcal{S}_{t-1,1,F}$  in more than one point. Now choose

- a subspace  $S_{s-t-1} \subset S_s$  such that  $S_{s-t-1} \cap \langle \mathcal{S}_{t-1,1,F} \rangle = \emptyset$ ;
- a Segre variety  $\mathcal{S}_{s-t-1,1,F} \subset \mathcal{S}_{s-1,1,F}$ , such that  $\langle \mathcal{S}_{s-t-1,1,F} \rangle \cap \langle \mathcal{S}_{t-1,1,F} \rangle = \emptyset$ ;
- two distinct  $A^I, B^I \in \mathcal{S}_{s-t-1,1,F}^I$ .

Since  $\langle \mathcal{S}_{s-t-1,1,F} \rangle$  and  $\langle \mathcal{S}_{t-1,1,F} \rangle$  are complementary subspaces of  $\langle \mathcal{S}_{s-1,1,F} \rangle$ , a projection map

$$\pi : \langle \mathcal{S}_{s-1,1,F} \rangle \setminus \langle \mathcal{S}_{t-1,1,F} \rangle \rightarrow \langle \mathcal{S}_{s-t-1,1,F} \rangle$$

is defined by  $\pi(P) = \langle P \cup \mathcal{S}_{t-1,1,F} \rangle \cap \langle \mathcal{S}_{s-t-1,1,F} \rangle$ .

Now suppose  $\pi(S_{s-t-1}) \cap \mathcal{S}_{s-t-1,1,F} = \emptyset$ . In  $\langle \mathcal{S}_{s-t-1,1,F} \rangle$  consider

- the regulus  $\mathcal{R}$  corresponding to  $\mathcal{S}_{s-t-1,1,F}^I$ , and the projectivity  $\kappa : A^I \rightarrow B^I$  such that, for any  $P \in A^I$ , the line  $\langle P, \kappa(P) \rangle$  belongs to  $\mathcal{S}_{s-t-1,1,F}^{II}$ ;
- the regulus  $\mathcal{R}'$  containing  $A^I$ ,  $B^I$  and  $\pi(S_{s-t-1})$ , and the projectivity  $\kappa' : A^I \rightarrow B^I$  such that, for any  $P \in A^I$ , the line  $\langle P, \kappa'(P) \rangle$  is a transversal line of  $\mathcal{R}'$ .

Since  $F$  is an algebraically closed field,  $\kappa'^{-1} \circ \kappa$  has a fixed point  $P$ . Therefore  $\kappa(P) = \kappa'(P)$ , so  $\mathcal{R}$  and  $\mathcal{R}'$  have a common transversal. This contradicts  $\pi(S_{s-t-1}) \cap \mathcal{S}_{s-t-1,1,F} = \emptyset$ . So, a point  $P \in S_{s-t-1}$  exists such that  $\pi(P) \in \mathcal{S}_{s-t-1,1,F}$ .

Next, let  $C^I \in \mathcal{S}_{s-1,1,F}^I$  be such that  $\pi(P) \in C^I$ , and  $Q$  the point in  $\langle \mathcal{S}_{t-1,1,F} \rangle$  such that  $Q$ ,  $P$ , and  $\pi(P)$  are collinear. If  $Q \in S_t$ , then  $\pi(P) \in S_s$ , a contradiction; also  $Q \in C^I$  leads to a contradiction (since it implies  $P \in C^I$ ). So  $Q \notin S_t \cup C^I$  and by a dimension argument two points  $Q_1 \in C^I \setminus S_t$  and  $Q_2 \in S_t \setminus C^I$  exist such that  $Q$ ,  $Q_1$  and  $Q_2$  are collinear: they are on the unique line through  $Q$  meeting both  $C^I \cap \langle \mathcal{S}_{t-1,1,F} \rangle$  and a  $(t-1)$ -subspace of  $S_t$  disjoint from  $C^I$ .

The plane  $\langle P, Q_1, Q_2 \rangle$  contains the lines  $PQ_2 \subset S_s$  and  $\pi(P)Q_1 \subset \mathcal{S}_{s-1,1,F}$  which meet outside  $\langle \mathcal{S}_{t-1,1,F} \rangle$ . This is again a contradiction.  $\square$